CONDITIONS TO THE DENSITY OF ACCESSIBLE SETS

DIEGO S. LEDESMA

Abstract. Given a control system $\dot{p} = X_0 + \sum_{i=1}^{k} u_i(t)X_i(p)$ on a compact manifold M we study conditions for the foliation defined by the accessible sets be dense in M. To do this we relate the control system to a stochastic differential equation and, using the support theorem, we give a characterization of the density in terms of the infinitesimal generator of the diffusion and its invariant measures. Also we give a different proof of Krener's theorem.

1. Introduction

This work addresses to the following problem: Let (M, g) be a compact Riemannian manifold without boundary and $F = \{X_0, X_1, \ldots, X_k\}$ be a family of vector fields over M. Consider the control system

$$\dot{p} = X_0 + \sum_{i=1}^{k} u_i(t)X_i(p), \tag{1}$$

with $u_i : \mathbb{R}_{\geq 0} \to \mathbb{R}$. Denote by $p_u(t)$ the solution of (1) for some

$$u = (u_1, \ldots u_k) \in C^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^k)$$

and such that $p_u(0) = p$. We call any such solution as a control path starting at p and denote by $\mathcal{CP}_p$ the set of all control paths starting at p.

Let $\mathcal{A}(p)$ be the accessible set from $p \in M$ defined by (see, for example, Agrachev [1] or Colonius and Kliemann [2])

$$\mathcal{A}(p) = \{p_u(t) \in M,\ t \geq 0\ e\ u \in C^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^k)\}.$$

It is well known that the accessible sets from a foliation with singularities (see P. Stefan [10]). In this work we want to find conditions for $\overline{\mathcal{A}(p)} = M$ for all $p \in M$. In order to give an answer to this question we will use stochastic methods. This can be justified by the support theorem for diffusions (see, for example, Kunita [9]).

The article is organized as follows: In section 2 we review the main tools of stochastic differential equations (SDE) that we will use. In section 3, given a set of vector fields we construct a diffusion, via a SDE, and study our problem by its properties. Finally in section 3 we change the point of view, given a set of vector field, we study the foliation induced by the attainable set and construct a diffusion that respect the foliation, then we give an answer to our problem in terms of the ergodic properties of this process. Also we give another proof of the Krener's theorem.

Key words and phrases. Control Systems, Diffusion Processes, Stochastic Calculus, Brownian Motion, Foliations

2010 *Mathematics Subject Classification. Primary: 60J60, 58E25; Secondary:60J70, 37N35, 60J65, 60J35.*

2. Stochastic Diferential Equations

Given a family $F = \{X_0, X_1, \dots, X_k\}$ of vector fields in M and a Brownian motion $B = (B_1, \dots, B_k)$ in $\mathbb{R}^k$ we construct a stochastic differential equation (SDE)

$$
\begin{aligned}
dp_t &= X_0(p_t)\ dt + \sum_{i=1}^{k} X_i(p_t)\ \circ dB^i \\
p_0 &= p.
\end{aligned}
\tag{2}
$$

A solution for this equation is an adapted stochastic process Y in M satisfying the following

$$
\begin{aligned}
f(Y_s) - f(p) &= \int_0^t X_0 f(Y_s)\ dt + \sum_{i=1}^{k} \int_0^t X_i f(Y_s)\ \circ dB_s^i \\
&= \sum_{i=1}^{k} \int_0^t X_i f(Y_s)\ dB_s^i + \int_0^t \left(X_0 + \frac{1}{2} \sum_{i=1}^{k} X_i^2 \right) f(Y_s)\ ds
\end{aligned}
$$

When the manifold is compact it is well known (see, for example, Elworthy [4] or N. Ikeda and S. Watanabe [6]) that there is a solution flow, this is a map $\phi : \mathbb{R}_{\geq 0} \times M \times \Omega \to M$ such that

i) $Y_t = \phi(t, p, \cdot)$ solves the SDE (2) with $Y_0 = p$, and
ii) for all f in $C^\infty(M)$ and $\omega \in \Omega$ we have that $f(\phi(t, \cdot, \omega))$ is a function in $C^\infty(M)$.

Associated to this solution flow there are the transition probabilities $\{P_t(p,\ \bullet\ ),\ p \in M,\ t \geq 0\}$ defined by

$$
P_t(p, U) = \mathbb{P}[\{\omega \in \Omega,\ \phi(t, p, \omega) \in U\}],
$$

for any Borel set U. It can be seen that its infinitesimal generator is given by

$$
\mathcal{L} = X_0 + \frac{1}{2} \sum_{i=1}^{k} X_i^2.
$$

Let $(\mathcal{W}(M) = \{s \in C([0, \infty), M)\},\ \mathcal{B}(\mathcal{W}(M)))$ be the path space of M furnished with the sigma algebra generated by the borel cylinders sets. Associated to the operator $\mathcal{L}$ there is a unique strongly Markovian system of probability measures $\{\mathbb{P}_p,\ p \in M\}$ defined for all borel set U in M by

$$
\mathbb{P}_p(s(t) \in U) = P_t(p, U)
$$

and satisfying

i) $\mathbb{P}_p[s \in \mathcal{W}(M),\ s(0) = p] = 1$
ii) $f(s(t)) - f(0) - \int_0^t (\mathcal{L}f)(s(r))\ dr$ is a $(\mathbb{P}_p,\ \mathcal{B}(\mathcal{W}(M)))$ martingale for any $f \in \mathcal{D}(\mathcal{L})$ e $p \in M$.

For a good reference see Ikeda and Watanabe [6, p. 202-205].

The transition probabilities generates a Markov semigroup $\{P_t, t \geq 0\}$ over $\mathcal{D}(\mathcal{L})$, by

$$
P_t f(x) = \int_M f(y)\ P_t(x, dy) \qquad \forall\ x \in M.
$$

a Borel measure μ over M is called invariant for the SDE (2) if

$$\int_M P_t f(x)\mu(dx) = \int_M f(x)\ \mu(dx)$$

or equivalently if

$$\int_M \mathcal{L}f(x)\ \mu(dx) = 0.$$

for all $f \in \mathcal{D}(\mathcal{L})$. Let

$$\mathcal{M} = \{\mu \in \mathcal{P}(M),\ \mu \text{ is invariant under } L\}.$$

An invariant measure μ is called ergodic if for any invariant set U, this is any set $U \subset M$ satisfying

$$\mathbb{P}[\{\omega,\ \phi(t,p,\omega) \subseteq U\ \forall\ t \geq 0\}] = 1 \qquad \text{for all } p \in U,$$

we have that $\mu(U)\mu(U^c) = 0$.

An important result for ergodic measures is the Ergodic Theorem (see, for example, Yosida [11]):

Theorem 1. *Let μ be an invariant measure and $f \in L^1(\mu)$ then, there is a function $f^* \in L^1(\mu)$ such that*

i) $f^* = \lim_{t\to\infty} \frac{1}{t}\int_0^t P_s f\ ds \qquad \mu - a.e.$

ii) $\int_M f^*(x)\ \mu(dx) = \int_M f(x)\ \mu(dx)$.

Also, when μ is ergodic and $f \in L^1(\mu)$ we have that

$$f^* = \int_M f\mu \qquad \mu - a.e.$$

From now on, we denote by $\mathcal{S}$ to the subset of M given by

$$\mathcal{S} = \bigcup_{\mu\in\mathcal{M},\ \text{ergodic}} \text{supp}(\mu) = \bigcup_{\mu\in\mathcal{M}} \text{supp}(\mu).$$

3. Control System

The relation between the control system (1) and the SDE (2) is given by the support theorem (see, for example, Ikeda and Watanabe [6] or Kunita [9]) that states

$$supp(\mathbb{P}_p) = \overline{\mathcal{CP}_p}$$

An easy consequence of this result is the following Lemma.

Lemma 1.

$$\mathcal{A}(p) \subseteq \bigcup_{t>0} supp(P_t(p,\cdot)) \subseteq \overline{\mathcal{A}(p)}, \qquad \forall\ p \in M.$$

Proof. To see the first inclusion we consider a point $q \in \mathcal{A}(p)$. Then, by definition, there will be a control path $p_u(s)$ such that $P_u(0) = p$ and $p_u(t) = q$. Let B_q be an open neighborhood of q and consider the open subset $U_t(B_q)$ of the path space defined by

$$U_t(B_q) = \{s \in C([0,\infty), M),\ s(0) = p,\ s(t) \in B_q\}.$$

Since $p_u(s) \in U_t(B_q)$ we get that $U_t(B_q)$ has non empty intersection with the set of all control paths $\mathcal{CP}_p$ starting at p and therefore, by the support theorem, we have that $\mathbb{P}_p[U_t(B_q)] > 0$. So $P_t(p, B_q) > 0$ and by the arbitrariness of B_q we get that $q \in supp(P_t(p,\ \bullet\ ))$.

For the second inclusion we assume that $q \in supp(P_t(p, \bullet))$ then for any open neighborhood B_q of q we have that $P_t(p, B_q) > 0$. Defining $U_t(B_q)$ as above we get that $\mathbb{P}_p[U_t(B_q)] > 0$ and therefore, by the support theorem we get that $U_t(B_q) \cap \mathcal{CP}_p \neq \emptyset$. So, there is a control path p_u such that $p_u(0) = p$ and $p_u(t) \in B_q$. By the arbitrariness of the open neighborhood B_q we get a sequence of points $\mathcal{A}(p)$ converging to q. Thus $q \in \overline{\mathcal{A}(p)}$.

□

From this result follows immediately that

Theorem 2. *Consider the control system (1). Then* $\overline{\mathcal{A}(p)} = M$ *for all* p *if and only if*

$$\int_0^\infty P_t(p, U) > 0$$

for all $p \in M$ *and open subset* $U \subset M$.

Proof. We assume that $\overline{\mathcal{A}(p)} = M$ for all $p \in M$. Then, for any open set U we have that

$$\left(\bigcup_{t \geq 0} supp(P_t(p, \bullet)) \right) \cap U \neq \emptyset$$

Thus, since the $P_t(p, \bullet)$ are continuous in t we get that there will be a small interval $I = (t_0 - \epsilon, t_0 + \epsilon)$ such that $P_t(p, U) > 0$ for any $t \in I$.

On the other side we assume that $\int_0^\infty P_t(p, U) > 0$ for all $p \in M$ and open set U of M and that there is a non dense $\mathcal{A}(p)$. Then picking $U = M \setminus \overline{\mathcal{A}(p)}$ we get a contradiction. □

Theorem 3. *The following assertions are equivalent*

i) $\overline{\mathcal{A}(p)} = M$ *for all* $p \in M$.
ii) *Every invariant measure* μ *of the diffusion given by equation (2) satisfy* $supp(\mu) = M$.
iii) $\{f \in C^2(\mathcal{S}) \cap \mathcal{D}(\mathcal{L}),\ \mathcal{L}f = 0 \text{ in } \mathcal{S}\} = \{f = const.\}$

Proof. To see that i) implies ii) we observe that if $p \in supp(\mu)$ then $supp(P_t(p, \bullet) \subseteq supp(\mu)$ for all $t \geq 0$. Then, $\mathcal{A}(p) \subseteq supp(\mu)$ by Lemma 1. Thus $M = supp(\mu)$.

For the converse, we assume that for any invariant measure μ we have that $supp(\mu) = M$ and consider the invariant measures μ_p defined by

$$\mu_p(\bullet) = \lim_{t \to \infty} \frac{1}{t} \int_0^t P_s(p, \bullet)\, ds.$$

since $supp(\mu_p) \subset \overline{\cup_{t \geq 0} supp(P_t(p, \bullet))}$ we get that $\overline{\mathcal{A}(p)} = M$

We will prove that $i)$ if and only if $iii)$. We assume that $\overline{\mathcal{A}(p)} = M$ for all $p \in \mathcal{S}$ and let $f \in C^2(\mathcal{S}) \cap \mathcal{D}(\mathcal{L})$ be a function such that $\mathcal{L}f = 0$. Then,

$$f|_{\mathcal{S}} = f^* = \lim_{t \to \infty} \frac{1}{t} \int_0^t P_s f\ dt$$

and, given an ergodic measure μ, the ergodic theorem implies that

$$f|_{\mathrm{supp}(\mu)} = \int_M f \mu \qquad \mu - \text{a.e.}$$

Since $\mathrm{supp}(\mu) = \overline{\mathcal{A}(p)}$ for some $p \in M$ we get that $\mathrm{supp}(\mu) = M$. Therefore $f = const.$

On the other side, we assume that

$$\{f \in C^2(\mathcal{S}) \cap \mathcal{D}(\mathcal{L}),\ \mathcal{L}f = 0 \text{ in } \mathcal{S}\} = \{f = const.\}$$

and that there is a $p \in M$ such that $\overline{\mathcal{A}(p)} \neq M$. Therefore, $\overline{\mathcal{A}(p)}$ is a compact invariant set in M. Consider a smooth function f such that $f = 0$ in $\overline{\mathcal{A}(p)}$ a $f = 1$ in the complement of an ϵ-neighborhood of $\overline{\mathcal{A}(p)}$ then $\mathcal{L}f^* = 0$ in M and therefore f^* must be constant. Since $\overline{\mathcal{A}(p)}$ is invariant and $Lf|_{\overline{\mathcal{A}(p)}} = 0$ we get that $f^*|_{\overline{\mathcal{A}(p)}} = 0$, so $f^* \equiv 0$. But then

$$\int_M f\ \mu = 0 \qquad \mu- \text{ a.e.}$$

for any ergodic measure μ. Since ϵ can be chosen as small as we want, we get that $\overline{\mathcal{A}(p)} = \mathcal{S}$. But, by a similar argument, if g is such that $g = 1$ in $\overline{\mathcal{A}(p)}$ and $g = 0$ in the complement of an ϵ-neighborhood of $\overline{\mathcal{A}(p)}$ then $\mathcal{L}g = 0$ in $\mathcal{S}$ but $g \neq const.$

□

Example 1. i- *Let F be a family of vector fields that spans the kernel of a submersion $f : M \to N$. Then for any non constant real function $g : N \to \mathbb{R}$ we have a non constant function $h = g \circ f$ satisfying $\mathcal{L}h = 0$. So we have points $p \in M$ such that $A(p)$ is not dense.*

ii- *Let M be a compact Riemannian manifold of dim(M)=3 with an orthonormal basis $\{X, Y, H\}$ of TM satisfying*

$$[X, H] = X \qquad [X, Y] = -H \qquad [H, Y] = Y.$$[1]

If $F = \{X, H\}$, consider the following SDE

$$dB = H\ \delta B^1 + X\ \delta B^2,$$

where (B^1, B^2) is the Brownian motion on $\mathbb{R}^2$. It is simple to see that $\mathcal{L} = \frac{1}{2}(X^2 + H^2)$. If f is a function such that $\mathcal{L}f = 0$. Since, div(X) = div(H) = 0 we get that the Lebesgue measure is invariant (by the Stokes Theorem). Then

$$2\int_M (|Xf|^2 + |Hf|^2)\ \mu_g = \int_M \mathcal{L}(f^2)\ \mu_g = 0$$

Also, if f is smooth we have that Yf is invariant by H. In fact,

$$\begin{aligned} H(Yf) &= Yf + YHf \\ &= Yf. \end{aligned}$$

So,

$$\begin{aligned} 2\int_M |Yf|^2 \mu_g &= 2\int_M (Yf) H(Yf) \mu_g \\ &= \int_M H(Yf)^2 \mu_g = 0. \end{aligned}$$

Then $Yf = 0$ and, therefore, f =const. So, the leaves of this foliation are dense. This result will be obtained later by a different point of view

[1] An example of this space can be obtained by the quotient of the universal covering of $SL(2, \mathbb{R})$ by a cocompact lattice

iii- *Let $S^n \subset \mathbb{R}^{n+1}$ the sphere and let F be the family defined by the vector field V given by the gradient of the height function $h(x_1, \dots, x_{n+1}) = x_1$, this is*

$$V(x) = (1, 0, \dots, 0) - x_1 x$$

for any $x \in S^n$. It is easy to see that the accessible sets are not dense. In this case, $\mathcal{S} = \{N, S\}$ and that $\mathcal{L} = \frac{1}{2}V^2$. Consider the function $f(x) = (1 - x_1^2)$ then $f \in C^2(\mathcal{S}) \cap C^2(M)$ and that $\mathcal{L}f = 0$ in $\mathcal{S}$ but f is non constant.

4. The General Case

Let F be a family of vector fields in M and let $\mathcal{D}_{Lie(F)}$ the associated distribution to the Lie algebra of F. We know that $\mathcal{D}_{Lie(F)}$ define a foliation in M which its integral manifolds are determined by the orbits of F.

Using the Nash embedding theorem it is possible to prove that, for N large enough, there is a family of vector fields $H = \{X_1, \dots, X_N\}$ such that, if $\pi(p) : T_pM \to \mathcal{D}_{Lie(F)}(p) \subset T_pM$ is the orthogonal projection and f is a smooth function, then

$$\nabla^D f(p) := \pi(p)(\nabla f(p)) = \sum_{i=1}^{N} (X_i f)(p) X_i(p),$$

and

$$g(\nabla^D f(p), \nabla^D f(p)) = \sum_{i=1}^{N} (X_i f)(p)^2.$$

With these vector fields we construct a SDE

$$dx_t = \sum_{i=1}^{N} X_i(x_t) \circ dB_t^i, \qquad x_0 = p, \tag{3}$$

whose solution is a diffusion with infinitesimal generator given by

$$\Delta = \frac{1}{2} \sum_{i=1}^{N} X_i^2.$$

Lemma 2. *The support of the transition probabilities associated to the SDE (3) are the leaves of the foliation given by $\mathcal{D}_{Lie(F)}$.*

Proof. This follows from the fact that the diffusion defined by (3) is a Brownian motion on the leaf of the associated foliation. Thus, the transition probability measures are completely continuous to the Lebesgue measure of the leaf. □

Once we have that the transition probabilities of the process defined by Eq. (3) are supported in the hole leaf we can use this process to give an answer of our problem. We do this in the following theorem where we also give another proof of Krener's Theorem ([8]).

Theorem 4. *The following assertions are equivalent*

i- *The leaves $\mathcal{A}(p)$ of the foliation defined by $\mathcal{D}_{Lie(F)}$ are dense in M*
ii- *Every invariant measure μ of the SDE (3) satisfies supp$(\mu) = M$*
iii- $\{f \in C^2(\mathcal{S}) \cap \mathcal{D}(\Delta),\ \Delta f = 0 \text{ in } \mathcal{S}\} = \{f = const.\}$

Moreover if $(\mathcal{D}_{Lie(F)})_p = T_pM$ for all p in M, then $\mathcal{A}(p) = M$ for all $p \in M$.

Proof. The first three assertions follow from Theorem 3 applied to the control system associated to equation (3).

For the last assertion we assume that $(\mathcal{D}_{Lie(F)})_p = T_pM$ for all p in M. The construction of the vector fields defining the equation (3) implies solution of the equation (3) is in this case the Brownian motion of the manifold starting at p. Then the support of the transition probabilities is the hole manifold. But, from Lemma 2 we get that support of the transition function are the leaves of the associated foliation. Therefore $\mathcal{A}(p) = M$ for all $p \in M$. □

Remark 1. *Theorem 4 can be applied in the context of a foliated manifold if we replace $\mathcal{D}_{Lie(F)}$ by the distribution that define the foliation.*

Example 2. i- *Consider the Torus given by $\mathbb{T}^2 = ([0,1] \times [0,1])/\sim$ where $(0,y) \sim (1,y)$ e $(x,0) \sim (x,1)$ and let X be the vector field defined by*

$$X = \partial_x + a\partial_y$$

The associated stochastic differential equation is given by

$$dx_t = X(x_t) \circ dB_t$$

for b_t a 1-d Brownian motion. The infinitesimal generator is

$$L = \frac{1}{2}\left(\partial_x^2 + a^2\partial_y^2 + 2a\partial_{xy}\right).$$

Then, if $a \in \mathbb{Q}$ we have that $a = m/n$ and

$$\mu = \sin^2(2\pi(mx - ny))\ dxdy$$

defines an invariant measure in $\mathbb{T}^2$. Therefore the leaves are not dense in this case, since it is possible to find a saturated set with non zero measure.

ii- *Consider again a compact Riemannian manifold M of dim(M)=3 with an orthonormal basis $\{X, Y, H\}$ of TM satisfying*

$$[X,H] = X \qquad [X,Y] = -H \qquad [H,Y] = Y.$$

Let $\mathcal{F}$ be a foliation induced by E =span$\{X, H\}$. The diffusion obtained by equation (3) is the Brownian motion on the leaves. It was shown by Garnett (see [5, Proposition 5]*, or* [3, section 3.4]*) that this diffusion has just one invariant measure which is the Lebesgue measure. Then we get again that the leaves of this foliation are dense.*

References

[1] A. Agrachev; Y. L. Sachkov. *Control theory from the geometric viewpoint.* Encyclopaedia of Mathematical Sciences, 87. Springer-Verlag, Berlin, 2004.

[2] F. Colonius; W. Kliemann *The dynamics of control.* Birkhuser Boston, Inc., Boston, MA, 2000.

[3] B. Deroin,V. Kleptsyn *Random conformal dynamical systems.* Geom. Funct. Anal. 17 (2007), Nro. 4, 10431105.

[4] K. D. Elworthy, *Stochastic Differential Equations on Manifolds.* London Math. Society (Lecture Notes Series 70) Cambridge University Press 1982

[5] L. Garnett, *Foliation, The ergodic theorem and Brownian motion.* Journal of Functional Analysis 51, (1983)pp. 285-311

[6] N. Ikeda and S. Watanabe *Stochastic differential equations and diffusion processes.* Second edition. North-Holland Mathematical Library, 24. North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, 1989.

[7] V. A. Kaimanovich,*Brownian motion on foliations: Entropy, invariant measures, mixing* Functional Analysis and Its Applications, Vol. 22, N^0 4, (1988) pp. 326-328
[8] A. J. Krener, *A generalization of Chow's theorem and the bang-bang theorem to non-linear control problems*. SIAM J. Control 12 (1974), pp. 43-52.
[9] H. Kunita, *Stochastic flows and stochastic differential equations*. Cambridge Univ. Press. (1997).
[10] P. Stefan. *Accessible Sets, Orbits and Foliations with Singularities*. Proc. London Math. Soc. (3) 29 (1974), pp. 699-713.
[11] K. Yosida,*Functional analysis*. Sixth edition. Springer-Verlag, Berlin-New York, 1980.